Question: Why is "Matrix **R**" important?

Significance of Matrix **R**

Review: Consider a spectral power distribution $N(\lambda)$ as a stimulus to vision. A set of color matching functions (CMFs) form the columns of a matrix A.

$$A = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ \vdots & \vdots & \vdots \end{bmatrix} . \tag{1}$$

Then the *fundamental metamer* $N^*$is the linear combination of the CMFs that is a least-squares best fit to N. Seeking a means to calculate N^*, Jozef Cohen found that $N^* = \mathbf{R}N$ when **R** is given by

$$\mathbf{R} = A(A'A)^{-1}A' . \tag{2}$$

Cohen derived Eq. (2) without knowing much of the mathematical context. At first he was pleased just to have an answer, a formula for the fundamental metamer. Later he learned that:
1. Matrix **R** of Eq. (2) is known to mathematicians as a projection matrix.
2. To find the fundamental metamer, the specific linear combination of CMFs, other methods exist. The Gram-Schmidt process can be applied to the columns of A, creating an orthonormal basis. It is then rather easy to derive N^* in terms of the orthonormal functions. A person familiar with orthonormal functions may consider it easier to use them. [Depending on how the orthonormal basis is used, it may use less computer memory, but in 2007 that is no longer important.]

So Cohen at first saw projection matrix **R** as a solution to one problem, finding the fundamental metamer of a light. Later, he saw that other methods could be used for that problem. But that is not the end of the story.

Locus of Unit Monochromats. It is easily proved that **R** is invariant with respect to the basis used in A. (Please see http://www.jimworthey.com/qna/invariance_of_r.html .) For instance, if we choose to work with the CIE's 2° observer, it can be expressed as $\bar{x}$, $\bar{y}$, $\bar{z}$, but also as cone functions, as an orthonormal basis, or as any other set of linear combinations of the initial functions. (The linear transformation must be reversible, involving a matrix whose inverse exists.) Then **R** is invariant in the strongest possible sense; it is the same big array of numbers in all cases. Since **R** is closely related to the locus of unit monochromats (LUM), the invariance of **R** implies that the LUM is an invariant shape for each observer. A different LUM will apply for the 10° observer, and a very different one may apply if the "observer" is a color camera, but in each case the LUM is an invariant embodiment of color matching for that observer.

Suppose that one begins by avoiding Matrix **R**, orthonormalizing the color matching functions, and then generating the LUM as a so-called parametric plot. That is, a narrow-band light of unit power and wavelength λ plots to a point in three-space $[\omega_1(\lambda), \omega_2(\lambda), \omega_3(\lambda)]$. Varying λ through the spectrum generates the LUM. However the orthonormal set is not unique. Others can be generated to embody the same observer. The invariance of **R** makes it clear that the *shape* of the LUM is invariant. Other brief algebra shows that the orthonormal basis generates the same LUM as **R** would create. Choosing a different orthonormal basis sets different axes, but does not alter the LUM's shape (except for a possible mirror inversion). Cohen liked to work directly from matrix **R** and to envision the LUM as floating in space without preferred axes; that approach emphasizes the invariant shape.

So, in short, Matrix **R** can serve as an important logical link in showing that the LUM is an invariant shape. Matrix

R can be part of the conceptual framework, even if the LUM is drawn by a different method.

Curve Fitting. Going in another direction, the projection matrix **R** is convenient for discussing practical calculations that may arise with color. A problem may be one that we think of as projecting functions into a subspace, but it may also be one that we think of as curve-fitting. If we have a function ϕ and we seek ϕ^*, which is the best fit to ϕ by a linear combination of any set of vectors A, then it is not necessary to set up and solve the least-squares problem in a series of steps. Just re-word the problem to say that ϕ is to be projected into the subspace of vectors A. Then compute the matrix **R** for the particular A, and write $\phi^* = \mathbf{R}\,\phi$. The concept of projecting ϕ into the column space of A is simple, and so is the computer calculation. In a matrix-oriented computer language, the formula for **R** can be expressed as one or two lines of code, or as a short function.

Curve Fitting Example. Word for word, here is a curve fitting example taken from the poster for CIC 14, the Color Imaging Conference in 2006:
“The more evolved procedure can be called ‘the fit first method.’ **The computer code looks like this:**

```
Rcam = RCohen(rgbSens)
CamTemp = Rcam*OrthoBasis
GramSchmidt(CamTemp, CamOmega)
```

Here `rgbSens` is a matrix whose columns are the 3 camera sensor functions. `Rcam` is Cohen’s projection matrix **R** based on the camera functions. `OrthoBasis` is **Ω**, the 3 orthonormal vectors for human. `CamTemp` is then the best fit to `OrthoBasis` using a linear combination of the camera sensitivities. The columns of `CamTemp` may not be orthonormal, so Gram-Schmidt finds the orthonormal basis, `CamOmega`. That’s the main result, and the camera’s LUM is a parametric plot of the 3 columns of `CamOmega`.”

The second line of the computer code does three curve fits! The short subroutines `RCohen()` and `GramSchmidt()` are available at http://www.jimworthey.com/omatrixcode.html .

Using matrix R in combination with the orthonormal basis. As explained at length elsewhere,
1. A set of orthonormal opponent color matching functions, **Ω**, has many interesting applications.
2. If an orthonormal basis **Ω** has been calculated, then the formula for **R** simplifies: $\mathbf{R} = \Omega\,\Omega'$, where the prime symbol indicates matrix transpose.
3. While the formula $\mathbf{R} = \Omega\,\Omega'$ *can* be used to compute the large numerical Matrix **R**, it can also be used in brief derivations of useful formulas. For example, one might want to convert a tristimulus vector in the **Ω** system to a tristimulus vector in the XYZ system. The needed conversion matrix can be derived by starting with $\mathbf{R} = \Omega\,\Omega'$.

Conclusions: When **Ω** is in use, **R** is not needed to generate color vectors, such as those that trace the locus of unit monochromats, but **R** may well be used for other best-fit problems that arise. The rows of **Ω** trace out the LUM, but the invariance of **R** reassures us that the LUM is an invariant shape.

In other words, the orthonormal basis and vectorial color can at times be used with little reference to matrix **R.** At other times, the projection matrix aids calculation or gives insight.
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