Question: Can we have fun with orthonormal functions?

Answer: Yes. Color matching data are legacy knowledge, standardized in 1931. Opponent colors is another old idea, and helps us to discuss color in intuitive terms. The orthonormal opponent model projects color stimuli into Cohen's space. As a further benefit, the orthonormal property makes possible some interesting shortcuts in deriving results that we need. Below is an appendix from the draft article on Vectorial Color. But first we need

Equations (11) and (12) from "Vectorial Color."

$$\mathbf{R} = \mathbf{A}[\mathbf{A}^{\mathrm{T}}\mathbf{A}]^{-1}\mathbf{A}^{\mathrm{T}} \tag{11}$$

$$|L^*\rangle = \mathbf{R}|L\rangle \quad . \tag{12}$$

Appendix D, Fun with Orthonormal Functions

This appendix is really about notation and convenient calculation. The ideas are the well-known facts of generalized Fourier series. Suppose that $\{|\omega_1\rangle, |\omega_2\rangle, |\omega_3\rangle\}$ are a set of functions that are orthonormal:

$$\langle\omega_i|\omega_j\rangle = \delta_{ij} \quad . \tag{8}$$

To be concrete we envision a set of 3 functions of wavelength, but there could be any number of functions over any domain. Now consider a function $|L\rangle$, which could be the spectral power distribution $L(\lambda)$ of a light. We want to approximate $|L\rangle$ by a linear combination of the functions $|\omega_i\rangle$:

$$|L\rangle \approx c_1|\omega_1\rangle + c_2|\omega_2\rangle + c_3|\omega_3\rangle \quad , \tag{D1}$$

where the c_j are constant coefficients. We seek a formula for c_1. Multiply Eq. (D1) on the left by $\langle\omega_1|$:

$$\langle\omega_1|L\rangle \approx c_1\langle\omega_1|\omega_1\rangle + c_2\langle\omega_1|\omega_2\rangle + c_3\langle\omega_1|\omega_3\rangle \; . \tag{D2}$$

By orthonormality, Eq. (8), $\langle\omega_1|\omega_2\rangle = \langle\omega_1|\omega_3\rangle = 0$, and $\langle\omega_1|\omega_1\rangle = 1$. Then $c_1 = \langle\omega_1|L\rangle$. In general,

$$c_j = \langle\omega_j|L\rangle \quad , \tag{D3}$$

and then substituting Eq. (D3) into Eq. (D1) leads to

$$|L\rangle \approx |\omega_1\rangle\langle\omega_1|L\rangle + |\omega_2\rangle\langle\omega_2|L\rangle + |\omega_3\rangle\langle\omega_3|L\rangle \quad . \tag{D4}$$

By reasoning not reviewed here, the sum on the right in Eq. (D4) is the linear combination of $|\omega_1\rangle$, $|\omega_2\rangle$, $|\omega_3\rangle$ that is the least-squares best fit to $|L\rangle$. That is also a description of the fundamental metamer of $|L\rangle$, denoted by $|L^*\rangle$. Therefore,

$$|L^*\rangle = |\omega_1\rangle\langle\omega_1|L\rangle + |\omega_2\rangle\langle\omega_2|L\rangle + |\omega_3\rangle\langle\omega_3|L\rangle \quad . \tag{D5}$$

(Why is '≈' gone in Eq. (D5)? Because the fundamental metamer *is* the approximation.) Factoring the

RHS of Eq. (D5) yields

$$|L^*\rangle = (|\omega_1\rangle\langle\omega_1| + |\omega_2\rangle\langle\omega_2| + |\omega_3\rangle\langle\omega_3|)|L\rangle \quad , \tag{D6}$$

or,

$$|L^*\rangle = \left(\sum_{j=1}^{3} \left|\omega_j\right\rangle\left\langle\omega_j\right|\right) |L\rangle \quad . \tag{D7}$$

The sum in parentheses is called a unity operator, $\mathbb{1}$, and could have N terms if there were N orthonormal vectors:

$$\mathbb{1} = \sum_{j=1}^{N} \left|\omega_j\right\rangle\left\langle\omega_j\right| \quad . \tag{D8}$$

Comparing Eq. (D7) to Eq. (12) shows that the unity operator performs the same function as projection matrix **R**, and suggests that it *is*[3] Matrix **R**. Recall that $|\omega_i\rangle$ is a column matrix and $\langle\omega_j|$ is a row matrix, so $|\omega_i\rangle\langle\omega_j|$ is a large square matrix. The summation from j=1 to N is separate from the matrix products, and indicates the sum of N large square matrices. So $\mathbb{1}$ is a large square matrix like **R**; it is not yet proved that they are equal.

Postponing that proof, why do we need a new symbol and formula for **R**, Eq. (D8)? The unity operator is not only an alternate formula for the projection matrix, it is a shorthand way to derive equations like Eq. (D4) or (D5), which include explicit formulas for the coefficients, as in Eq. (D3). Now letting N=3, we notice an alternate way of writing Eq. (D8):

$$\mathbb{1} = \begin{bmatrix} |\omega_1\rangle & |\omega_2\rangle & |\omega_3\rangle \end{bmatrix} \begin{bmatrix} \langle\omega_1| \\ \langle\omega_2| \\ \langle\omega_3| \end{bmatrix} \quad . \tag{D9}$$

In this case, the summation implicit in the matrix product is the one that was written explicitly in Eq. (D8). It is natural to write the orthonormal set as the columns of a matrix $\mathbf{\Omega}$, that is

$$\mathbf{\Omega} = [|\omega_1\rangle \; |\omega_2\rangle \; |\omega_3\rangle] \quad . \tag{D10}$$

Therefore

$$\mathbb{1} = \mathbf{\Omega}\,\mathbf{\Omega}^{\mathrm{T}}. \tag{D11}$$

Now if projection matrix **R** is based on color matching functions, it is the same for any transformed set, so substitute $\mathbf{A} = \mathbf{\Omega}$ in Eq. (11):

$$\mathbf{R} = \mathbf{\Omega}[\mathbf{\Omega}^{\mathrm{T}}\mathbf{\Omega}]^{-1}\mathbf{\Omega}^{\mathrm{T}} \quad . \tag{D12}$$

But the grouping $\mathbf{\Omega}^{\mathrm{T}}\mathbf{\Omega}$, because of orthonormality, is the 3×3 identity matrix:

$$\mathbf{\Omega}^{\mathrm{T}}\mathbf{\Omega} = \mathbf{I}_{3\times 3} \quad , \tag{D13}$$

whose inverse is also the identity matrix, therefore

$$\mathbf{R} = \boldsymbol{\Omega}\boldsymbol{\Omega}^{\mathrm{T}} \quad . \tag{D14}$$

Comparing Eq. (D11) to Eq. (D14) confirms that $\mathbf{R} = \mathbb{1}$.

The key idea of this appendix is contained in Eq. (D4) or (D5), or in Eq. (D8) or (D9), which are tools for deriving Eq. (D4) or (D5). Eq. (D3) is the explicit formula for the coefficients. If the summation notation of Eq. (D8) seems awkward, Eq. (D9) can be used to derive formulas. For example, $|L^*\rangle = \mathbb{1}|L\rangle$, then

$$|L^*\rangle = \begin{bmatrix} |\omega_1\rangle & |\omega_2\rangle & |\omega_3\rangle \end{bmatrix} \begin{bmatrix} \langle\omega_1| \\ \langle\omega_2| \\ \langle\omega_3| \end{bmatrix} |L\rangle \quad . \tag{D15}$$

On the RHS, three matrices are multiplied. Formally multiplying the second and third matrices gives

$$|L^*\rangle = \begin{bmatrix} |\omega_1\rangle & |\omega_2\rangle & |\omega_3\rangle \end{bmatrix} \begin{bmatrix} \langle\omega_1|L\rangle \\ \langle\omega_2|L\rangle \\ \langle\omega_3|L\rangle \end{bmatrix} \quad . \tag{D16}$$

Formal multiplication in Eq. (D16) then gives Eq. (D5), the desired result. A succinct insight is that $\boldsymbol{\Omega}^{\mathrm{T}}\boldsymbol{\Omega} = \mathbf{I}_{3\times3}$, Eq. (D13), but if the order of multiplication is reversed, $\boldsymbol{\Omega}\boldsymbol{\Omega}^{\mathrm{T}} = \mathbf{R}$, Eq. (D14).

Application. Suppose that we seek the relationship between the orthonormal vectors, $\boldsymbol{\Omega}$, and Guth's opponent functions (renormalized as in Fig. 1d). Call the array of Guth's vectors $\mathbf{G}$. Then

$$\mathbf{G} = \mathbb{1}\mathbf{G} \quad . \tag{D17}$$

In Eq. (D17), there is equality and not approximate equality because we know that the Guth color matching functions are linear combinations of the columns of $\boldsymbol{\Omega}$. Apply Eq. (D11):

$$\mathbf{G} = \boldsymbol{\Omega}\,\boldsymbol{\Omega}^{\mathrm{T}}\,\mathbf{G} \quad . \tag{D18}$$

A realistic situation is assumed: that $\mathbf{G}$ and $\boldsymbol{\Omega}$ exist on a computer as arrays of numbers. It might be that $\boldsymbol{\Omega}$ was just found from $\mathbf{G}$ by the Gram-Schmidt algorithm. We now seek a 3×3 matrix that is the transform from one to the other. All that we need to do is group the terms in Eq. (D18). Define $\mathbf{X} = \boldsymbol{\Omega}^{\mathrm{T}}\,\mathbf{G}$. Then $\mathbf{G} = \boldsymbol{\Omega}\,\mathbf{X}$, and

$$\boldsymbol{\Omega} = \mathbf{G}\,\mathbf{X}^{-1}. \tag{D19}$$

The inverse may be the more interesting. Numerically,

$$\mathbf{X}^{-1} = \begin{bmatrix} 1 & 0.3433 & 0.6442 \\ 0 & 1.0573 & 0.2706 \\ 0 & 0 & 1.1726 \end{bmatrix} \quad . \tag{D20}$$

We can then see that the first vector of $\boldsymbol{\Omega}$ is the same as the first vector of $\mathbf{G}$. The second vector of $\boldsymbol{\Omega}$ is a combination of the first 2 vectors in $\mathbf{G}$, the ones that depend only on red and green cones, and $\boldsymbol{\Omega}$'s third vector is a combination of all the Guth vectors. The same approach, beginning with Eq. (D17), can be used to find other relationships, such as **Ω** in terms of $\bar{x}, \bar{y}, \bar{z}$. To emphasize individual functions, Eq. (D9) or (D8) can be used for the unity operator. To begin,

$$\begin{bmatrix} \langle \bar{x}| \\ \langle \bar{y}| \\ \langle \bar{z}| \end{bmatrix} = \begin{bmatrix} \langle \bar{x}| \\ \langle \bar{y}| \\ \langle \bar{z}| \end{bmatrix} \mathbb{1} \quad . \tag{D21}$$

Then use Eq. (D9) for the unity operator and do one formal matrix multiplication, to obtain

$$\begin{bmatrix} \langle \bar{x}| \\ \langle \bar{y}| \\ \langle \bar{z}| \end{bmatrix} = \mathbf{T} \begin{bmatrix} \langle \omega_1| \\ \langle \omega_2| \\ \langle \omega_3| \end{bmatrix} \tag{D22}$$

where

$$\mathbf{T} = \begin{bmatrix} \langle \bar{x}|\omega_1\rangle & \langle \bar{x}|\omega_2\rangle & \langle \bar{x}|\omega_3\rangle \\ \langle \bar{y}|\omega_1\rangle & \langle \bar{y}|\omega_2\rangle & \langle \bar{y}|\omega_3\rangle \\ \langle \bar{z}|\omega_1\rangle & \langle \bar{z}|\omega_2\rangle & \langle \bar{z}|\omega_3\rangle \end{bmatrix} \quad . \tag{D23}$$

Now let Eq. (D22) be multiplied on the right by any light $|L\rangle$, then the result is a relationship for the tristimulus vectors of the light in the two systems:

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \mathbf{T} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \quad . \tag{D24}$$

The numerical matrix is T = $\begin{bmatrix} 6.449 & 4.929 & 2.468 \\ 8.787 & 0 & 0 \\ 0.968 & -1.903 & 11.655 \end{bmatrix}$. Eq. (D24) is the conversion from a tristimulus vector in the orthonormal schema to one in the legacy system. In Eq. (D23), a square matrix is shown to emphasize that the result will be an array of 9 numbers. For the computer calculation, the matrix can be left factored out:

$$\mathbf{T} = \left[|\bar{x}\rangle \quad |\bar{y}\rangle \quad |\bar{z}\rangle \right]^T \mathbf{\Omega} \quad . \tag{D25}$$