Friday, 2014 November 21, 22:51:02
JAW, Sunday, 2006 January 1, 23:20:48

Applications of Vectorial Color

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Abstract

A companion article introduced a set of orthonormal opponent color matching functions. That "orthonormal basis" is an expedient for plotting lights in Jozef Cohen's logical color space. Indeed, graphing the new color matching functions (CMFs) together (as a parametric plot) gives Cohen's invariant "locus of unit monochromats," or LUM. In this article, the functions and related vector methods are applied to fundamental problems. In signal transmission and propagation-of-errors work, it is desirable to describe stimuli by decorrelated components. The orthonormal CMFs inherently do this, and an example is worked out using a large set of color chips. Starting with the orthonormal functions, related functions, such as cone sensitivities, are graphed as directions in color space, showing their intrinsic relationships. Building on work of Tominaga and co-authors, vectorial plots are related to the problem of guessing the illuminant, a step towards a constancy method. The issue of color rendering is clarified when the vectorial compositions of test and reference lights are graphed. A single graph shows the constraint that the total vectors are the same, but also shows the differences in colorimetric terms. Since the LUM summarizes a trichromatic system by a 3-dimensional graph, dichromatic observers can be represented by 2-D graphs, revealing details in a consistent way. The "fit first method" compares camera to human, applying the Maxwell-Ives criterion in graphical detail.
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< The figures are in another file: http://www.jimworthey.com/appsvectorialfigs.pdf >

Introduction

Background. A companion article[1] introduces a set of orthonormal opponent color matching functions, which for short can be called the orthonormal basis. The orthonormal basis can be a transform of the CIE's 2-degree or 10-degree observer[2,3], although the ideas transcend the choice of standard observer. In traditional practice, color vectors $[X\ Y\ Z]^T$ are added numerically but seldom graphed or discussed, presumably because of the arbitrariness in the XYZ scheme. When the orthonormal basis is used, vectors plot into Jozef Cohen's intrinsically more logical color space. In effect, the research of Jozef Cohen, William A. Thornton, Michael H. Brill, Sherman Lee Guth, Worthey and others bears fruit in the set of three functions.[1] It is not that the functions evoke the personalities of the researchers, their big words and quirky modes of expression, but the opposite. Decades of research at last allow us to take the personalities and arbitrariness **out** of the color mixing discussion. Color stimuli add linearly, which sounds simple, but the overlap among cone sensitivities must be dealt with. In color work, the issue of overlap is confronted by opponent colors, but also by Cohen's Matrix R.[4-6] In mathematics, an orthonormal basis is a general means for working with a set of linearly independent functions. Combining all these ideas leaves little room for arbitrary features, but fosters clear discussion. In this article, the orthonormal basis and vectorial methods will be applied to some interesting problems.

Method. The orthonormal functions are an achromatic (whiteness) function, a red-versus-green function, and a kind of blue-versus-yellow function that is not very different from just blue sensitivity, Fig. 1. Cohen's invariant color space thus gains axes with color names: whiteness; red or green; and blue or yellow. The functions can be called $\omega_1(\lambda)$, $\omega_2(\lambda)$, $\omega_3(\lambda)$ or they can be the columns of a matrix $\mathbf{\Omega}$:

$$\mathbf{\Omega} = [|\omega_1\rangle\ |\omega_2\rangle\ |\omega_3\rangle] \quad . \tag{1}$$

The ket notation, such as $|\omega_1\rangle$, makes explicit that the functions are being written as column vectors. The achromatic function ω_1 is proportional to the familiar $\bar{y}$ and is considered to be a sum of red and green cone sensitivities (with certain coefficients). Function ω_2 is a difference, red cones minus green cones, such that it is orthonormal with ω_1. The third function has contributions from all three cone types, and completes the orthonormal set. That is

$$\langle\omega_i|\omega_j\rangle = \delta_{ij} \quad , \tag{2}$$

where $\delta_{ij} = 1$ if i=j, 0 otherwise. The bra form, $\langle\omega_i|$, is a row vector, so $\langle\omega_i|\omega_j\rangle$ is an inner product, which one could also write as $\sum_{\lambda=360}^{830} \omega_i(\lambda)\omega_j(\lambda)$. The bracket notation suppresses the variable of summation (or integration) λ, and the limits.

As with any color matching functions, one may find each function's inner product with the SPD of a light L, yielding the tristimulus vector V of the light:

$$V = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \mathbf{\Omega}^{\mathrm{T}}|L\rangle \quad . \tag{3}$$

That is to say, the 3-vector V is a summary description of the radiometric function L, which might be a 471-vector expressing radiance versus wavelength. In the legacy color algebra, the tristimulus vector is $[X\ Y\ Z]^{\mathrm{T}}$.

Applications

Correlation of variables. One benefit of an opponent color step is that its outputs are less correlated than the receptor signals, more suited for efficient use of a communications channel.[7] In color metrology, less correlated color signals make it more practical to estimate propagation of errors.[8,9]

We can see by examples how the orthonormal basis expresses stimuli in comparatively decorrelated vector components. García-Beltrán, Nieves, Hernández-Andrés, and Romero calculated "Linear Bases for Spectral Reflectance Functions of Acrylic Paints," and as a starting point measured spectral reflectances of numerous paint samples prepared by a Dr. Eloisa Jiménez Martín.[10] Dr. García kindly sent to me as computer files the spectral reflectances of

5572 samples, measured from 400 nm to 700 nm, at 1 nm intervals. The authors sought to have a "gamut of spectral reflectance curves as broad and varied as possible in order to cover the greatest range of observed variation in the functions of opaque objects that surround us."[10] They classified the samples into 5 hue categories: red, yellow, green, blue, and purple, with 1080, 1062, 1207, 1063, and 1160 samples, respectively, as I received the data.

A varied set of object spectral reflectances is thus available. For an example of correlated measurements, let red and green cone stimuli be calculated according to human cone sensitivities, with D65 as the light, for each of the 5572 samples. In Fig. 2, each dot corresponds to a paint chip, plotted according to its cone stimuli, *R* and *G*. The red, green, blue, and purple groups are plotted in their nominal colors; the dots for the yellow group are plotted in maroon. The correlation coefficient of *R* and *G* is 0.976, meaning that they are highly correlated. The formula for correlation coefficient is symmetrical in the two variables; it does not matter which is named first. In Fig. 2, the stimuli cover a region near a straight line and are clearly correlated.

Fig. 3 shows a plot with the same paint samples and light D65, but now v_2 is graphed vs v_1 , based on the orthonormal functions ω_2, ω_1, according to Eq. (3). Now the correlation coefficient is 0.180, and the points are visibly more spread out. Although correlation has been reduced, the joint variation of v_1 and v_2 is still constrained by a rough gamut boundary.

Fig. 4 moves to the XYZ system. The object colors and light remain the same, but *X* is graphed versus *Y*. The correlation coefficient is found to be 0.960. Table 1 lists the correlation coefficients and direction cosines for 9 pairings. Notice that direction cosine is a simple result involving only two sensitivity functions, whereas correlation coefficient is a statistic involving the paint chips and the light, along with two sensitivities.

Table 1. Correlation coefficient and direction cosine for various pairings of functions. The correlation coefficients are based on D65 and the paint samples as in Figs. 1-3. Direction cosine is a measure of overlap between the sensitivity functions.

Functions compared	**Correlation Coefficient**	**Direction Cosine**
R cones, G cones	0.976	0.918
ω_1, ω_2	0.180	0
$\bar{x}$, $\bar{y}$	0.960	0.760
R cones, B cones	0.520	0.058
G cones, B cones	0.619	0.121
$\bar{x}$, $\bar{z}$	0.548	0.255
$\bar{y}$, $\bar{z}$	0.557	0.082
ω_1, ω_3	0.522	0
ω_2, ω_3	−0.303	0

The three function pairs compared graphically and above the heavy line in Table 1 are the most interesting and somewhat comparable. In nature, the *R* and *G* cones highly overlap, and the B cone sensitivity is by comparison isolated in the short-wavelength end of the spectrum. (Night vision sensitivity falls neatly into the gap between *B* and *G* cones, but that is far off the topic here. The topic is how to make better use of established knowledge about trichromatic color vision.) In the orthonormal system, stimulus values v_1 and v_2 are by design an alternate presentation of the information in *R* and *G*. In the XYZ system, *Z* is the blue stimulus, so *X* and *Y* have the *R* and *G* information.

The direction cosine of two vectors *f* and *g* is defined by

$$\text{direction cosine} = \langle f|g\rangle/(\langle f|f\rangle\langle g|g\rangle)^{1/2} \quad . \tag{4}$$

For vectors in 3-space, Eq. (4) agrees with the usual idea of the cosine of the angle between vectors, but it applies also to vectors of higher dimension, such as functions of wavelength.

In short then, opponent-color systems are expected to provide a decorrelated representation of red-green information, and the orthonormal system serves this purpose. By working with a set of measured data we get a practical demonstration and minimize theoretical discussion. I do not claim that the orthonormal basis is in some way optimum for decorrelating stimulus values, as evaluated by the correlation coefficient. That would be hard to prove because the calculation depends on the specific paint samples and the light. We can say that the orthonormal basis spreads out stimulus vectors as much as possible, and the XYZ system by comparison squeezes the vectors together. Figs. 2-4 can be thought of as projections into certain planes of a set of tristimulus vectors, and in particular of the red-green information. Figure 3, which spreads out the red-green information, agrees best with the normal concept of graphing. If one dot corresponds to a red chili pepper and another to a green bell pepper, the difference is apparent.

Propagation of Errors. When measured values are added or otherwise combined, reduced correlation of the vector components makes it more practical to estimate the error in the result. If the errors are considered independent, then they combine as the square root of the sum of the squares, or some other simple formula. If the errors are covariant, more complicated formulas apply and covariances must be estimated.[11] It is desirable to assume that errors are independent, even when it is not perfectly true. Tristimulus vectors found from the orthonormal basis have reduced covariances among the components, especially between red and green.

David MacAdam asserted that orthonormal CMFs give uncorrelated errors under certain conditions[8]. Starting with cone sensitivities, Buchsbaum and Gottschalk found transformed signals for optimum information transmission, a set of orthonormal CMFs[7]. The analysis above is intended to have a more practical flavor. The orthonormal basis has various attractive qualities and was already defined[1]. MacAdam's and Buchsbaum's general ideas are tested above on real data. MacAdam suggested in 1953 that "some use for orthogonal mixture functions may be found" in color television, and indeed the NTSC color television standard used a kind of opponent-color system[12]. More generally, the idea of efficient signal transmission[7] applies to image compression. The new basis **Ω**–in fact very similar to that found in Ref. 7–could be used as a starting point in compressing color images.

MacAdam's orthonormal functions, like **Ω**, are based on $\{\bar{x}, \bar{y}, \bar{z}\}$. We now know, based on Cohen's work[4-6], that any CMFs equivalent to an initial set lead to the same projection matrix **R**, and therefore to the same locus of unit monochromats, LUM. That is, if the LUMs are considered to have wavelength markings, and the axes are ignored, then the three-dimensional shapes will be congruent. (If one is a mirror-image of the other, a minus sign can be introduced.) Buchsbaum started with Vos-Walraven cone functions[7], but still the "Buchsbaum LUM," would be similar.

Color matching functions mapped to 3-vectors. The previous article's Fig. 1[1] illustrates alternate sets of color matching functions that are linear combinations of a given set, such as the 2° observer. Alternate sets predict the same color matches, but differ according to further meaning, whether cone sensitivities, CIE primaries, or something else. Suppose that C is such a set of CMFs, known to be a linear combination of $|\omega_1\rangle$, $|\omega_2\rangle$, $|\omega_3\rangle$. That is,

$$C = [|c_1\rangle\ |c_2\rangle\ |c_3\rangle]\ , \tag{5}$$

related to **Ω** by

$$C = \mathbf{\Omega}\, B\ , \tag{6}$$

where B is a 3×3 transform matrix. Multiply Eq. (6) on the left by $\mathbf{\Omega}^{\mathrm{T}}$. By orthonormality, $\mathbf{\Omega}^{\mathrm{T}}\mathbf{\Omega}$ is an identity matrix, then

$$B = \mathbf{\Omega}^{\mathrm{T}} C\ . \tag{7}$$

We then notice that

$$B = \begin{bmatrix} \langle\omega_1| \\ \langle\omega_2| \\ \langle\omega_3| \end{bmatrix} \begin{bmatrix} |c_1\rangle & |c_2\rangle & |c_3\rangle \end{bmatrix} = \begin{bmatrix} \langle\omega_1|c_1\rangle & \langle\omega_1|c_2\rangle & \langle\omega_1|c_3\rangle \\ \langle\omega_2|c_1\rangle & \langle\omega_2|c_2\rangle & \langle\omega_2|c_3\rangle \\ \langle\omega_3|c_1\rangle & \langle\omega_3|c_2\rangle & \langle\omega_3|c_3\rangle \end{bmatrix}\ . \tag{8}$$

For numerical calculation, Eq. (7) would suffice, but Eq. (8) has an interesting property. The 3 columns of B are the tristimulus vectors of the 3 vectors $|c_j\rangle$, or would be if the sensitivities $|c_j\rangle$ were lights. These column vectors can be plotted to visualize the relationships among CMFs.

Fig. 5 presents an example. The basis functions plot to the axes themselves since, for example,

$$\mathbf{\Omega}^{\mathrm{T}}\,|\omega_1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\ , \tag{9}$$

by orthonormality. Since $\bar{y}$ is proportional to ω_1, it also plots to the v_1 axis. (This one discussion ignores vector lengths.) Red and green cones lie in the ω_1-ω_2 plane. $\bar{x}$ is out of that plane. Blue cones are labeled with a simple 'b,' which is not legible but overprinted on the notation z-bar, since those functions are proportional. Fig. 5's pictorial presentation is complemented by numerical details such as the direction cosines of Table 1. Fig. 5 shows intuitively which functions are similar or not similar. For example, the direction cosines in Table 1 are computed directly by Eq. (4), but the same results could be computed from the 3-vectors underlying Fig. 5.

Importance of orthonormality. The statements about Fig. 5 do not apply if the starting functions are a non-orthogonal set, such as $\{\bar{x}, \bar{y}, \bar{z}\}$. Then for example, $\bar{x}$ does not plot to the X axis and cosines between 3-vectors are not the cosines between the original functions. When the orthonormal CMFs are used, the simpler rules apply: ω_1 plots to the v_1 axis, and the cosine between two vectors in the diagram is the same as the cosine between the original functions. In fact, if c and d are CMFs, the inner product of the functions equals that of the corresponding 3-vectors. Recall that with the orthonormal basis, the tristimulus values are the coefficients in the orthonormal function expansion of a function:

$$|c\rangle = c_1|\omega_1\rangle + c_2|\omega_2\rangle + c_3|\omega_3\rangle \ , \tag{10}$$

$$|d\rangle = d_1|\omega_1\rangle + d_2|\omega_2\rangle + d_3|\omega_3\rangle \ , \tag{11}$$

where $c_j = \langle c|\omega_j\rangle$ and $d_j = \langle d|\omega_j\rangle$. Then

$$\langle d|c\rangle = (\ d_1\langle\omega_1| + d_2\langle\omega_2| + d_3\langle\omega_3|\ )\ (\ c_1|\omega_1\rangle + c_2|\omega_2\rangle + c_3|\omega_3\rangle) \ . \tag{12}$$

The product on the right contains 9 terms, but by orthonormality, Eq. (2), 6 of them are zero, and

$$\langle d|c\rangle = d_1c_1 + d_2c_2 + d_3c_3 \ , \tag{13}$$

which is the inner product of the 3-vectors. The direction cosine, Eq. (4), is based on inner products.

Lighting and color

If two white lights have the same total tristimulus vector, but markedly different spectral composition, then they are said to differ in *color rendering*. Making a transition from spectral composition to *vectorial* composition reveals differences between the lights, and that method was illustrated in the earlier article.[1] To make a fresh start here, let us review what color rendering is **not**.

Guessing the illuminant. Vrhel *et al.*[13] measured spectral reflectances of 64 Munsell Colors including 12 neutrals. Let the 64 chips be illuminated by two blackbody spectra, first 5000 K, then 6500 K. The two lights are set to equal values of v_1, the achromatic stimulus component. (In other words, they are equated for illuminance on the chips.) When the light changes, the tristimulus vector of each colored paper changes, Figs. 6, 7 and 8. The lightest neutral, N9.5, is indicated on each drawing and serves as a proxy for the light itself. A virtual reality 3D graph may also be available on the web site.[14] The vector shifts are the *physical substrate* for any potential color constancy mechanism. That is, a change of lighting occurs and the figures show the resulting changes in the 64 stimuli, which a constancy mechanism would be intended to reverse[15]. Tominaga, Ebisui and Wandell studied the color effects of blackbody lights at different temperatures[16,17]. Using stimulus vectors based on camera sensors, they found the temperature shift to be best revealed by an object-color gamut in a red-blue plane. The details were specific to the camera used, but the idea is more general.

Using the invariant space, object color gamuts or averages could be worked out in the v_2-v_3 plane, Fig. 8, or v_3-v_1, Fig. 7. The key idea is to project the vectors into a plane where the increase of blue stimulation with increasing color temperature becomes evident. In Fig. 8, the locus of blackbody light at constant v_1 is indicated by circles, labeled with Kelvin temperatures.

To restate, Tominaga *et al.*[16,17] saw the value of color vectors, in preference to other measures of stimuli. Mapping colors of a scene to vectors, then projecting into a plane with blue on one axis, revealed a marked gamut shift with changing temperature of a blackbody illuminant. Projecting gamuts into (x, y) was not helpful.[16,17] Vectors V computed with the orthonormal basis can also reveal the blueness, and therefore the color temperature, of a light. The section below on camera analysis shows how electronic sensitivities can be treated similarly to human ones.

Blackbody plane. A special plane can be defined for tracking the effects of blackbody temperature. The blackbody locus (at constant radius, for example) in the 3D invariant space is not parallel to the v_1-v_2 or v_2-v_3 plane, but does lie close to a plane through the origin, especially for temperatures above 2000 K. Color vectors from the origin for the 2 blackbodies at 3000 K and 10000 K can be used to define a blackbody plane. As one blackbody light is substituted for another, object colors will move roughly parallel to this plane, and projecting them into the plane will give a 2D picture with a good view of the light's effects. In that presentation, algorithms for recovering the light can be visualized and tested. Using the algebra of Appendix A, the 64 Munsell colors are plotted in that plane, Fig. 9. The colors of the lights are essentially the tail and head of the arrow for N9.5. The heavy black arrow indicates the mean stimuli under the 2 lights. The dashed lines then project those colors toward the blackbody locus, showing that average chip color is at best an indirect measure of light color. Results for actual scene data may differ; the point here is that a change in blackbody temperature gives systematic object color shifts, well visualized in the invariant space or projected into a "blackbody plane."

Color temperature is usually considered a dimension of normal variation among white lights, consistent with the picture of object colors marching in formation in Fig. 9, and with the variation among reference illuminants in the Color Rendering Index" document[18]. So-called "color rendering" then is the issue of white lights whose variation is abnormal. Such a light can have the same tristimulus vector as a normal light—blackbody or daylight—but a different vectorial composition. One could say "different spectral composition," but a vectorial approach keeps colorimetry in the discussion.

Color rendering. For a narrow-band light of unit power (a "unit monochromat") and wavelength λ, there is a vector $\boldsymbol{\omega}(\lambda)$ of fixed direction and amplitude. Those vectors are the rows of $\boldsymbol{\Omega}$ and trace the LUM. A more general light has a power distribution $P(\lambda)$. For the narrow band centered at λ, the vector is $P(\lambda)\boldsymbol{\omega}(\lambda)$. Symbol $\boldsymbol{\omega}$ (lowercase omega) is bold, to emphasize that $P(\lambda)$ is a scalar but $\boldsymbol{\omega}(\lambda)$ is a vector with length and direction in color space. If 10-nm bands are used, then for each 10 nm there is a vector $P(\lambda)\boldsymbol{\omega}(\lambda)$ representing that band's contribution to the light's total tristimulus vector. In simple terms, a light has about 30 degrees of freedom, since (700-400)/10 = 30, meaning 30 vector *amplitudes*. The directions of the small vectors can vary, but only a little. The light's total tristimulus vector has 3 degrees of freedom. Most lights of poor color rendering fall short in similar ways, but treating a light as a sum of about 30 vectors *does not pre-judge the situation*. Looking at vectorial composition applies colorimetry to each wavelength band, then compares the reference and test lights in detail. Starting at short wavelengths, the vector components of each light are added tail-to-head[1], confirming that test and reference give similar total vectors, but take different paths to the total.

(It is admittedly a little confusing to focus on the rows of $\mathbf{\Omega}$ as being 3-vectors. In effect, we are looking at the usual calculation of tristimulus vectors, and re-arranging the order of summation. Re-arranging the order of sums is one of the basic tricks of applied math, and a benefit of thinking in vector terms.)

The previous article had one example of a mercury vapor light compared to JMW daylight[1]. A figure showed a view of a 3-dimensional graph. Problems that occur are mainly a matter of reds and greens[1,19,20], making the v_1-v_2 plane appropriate for a flat projection. Fig. 10 again shows the vectorial composition of the same mercury light versus daylight, but now projected into v_1-v_2. In Fig. 11, the 64 Munsell chips[13] are plotted in the invariant space, then projected into v_1-v_2. The chips lose redness ($v_2 > 0$) or greenness ($v_2 < 0$) and crash towards neutral[19]. Fig. 11 tells the main story, but the 3D picture and other flat views could be generated. Nothing forces us to lose information.

Color vision defects

Fig. 5 and the related discussion show that the 3 cone pigments have specific directions in color space. When a person has two normal cone systems, but lacks one type, then the remaining receptor systems define a plane. The trichromatic LUM can be projected into that plane using the method of Appendix B, giving a mapping of the spectrum for each type of defect, indicated by the heavy black lines in Figs. 12-14. (Please ignore the thinner lines at first.) Within the stated method, plus and minus axis directions were chosen for convenience. Where the dichromatic LUM approximates a straight line from the origin, we expect poor wavelength discrimination and without belaboring the details, that is consistent with data[21]. Alternatively, an orthonormal basis could be generated directly from the sensitivities of the color defective's remaining cones, giving the same 3 LUMs, but different axes.

Now consider the thinner curves in Figs. 12-14, which have their own wavelength markings. They represent a chain of narrow-band stimuli being summed for the equal-energy light. In Fig. 10 and elsewhere, the chain of vectors was shown with arrowheads and 10-nm wavelength intervals. Now the intervals are 1 nm, and arrowheads are absent. The summed chain gives the vector of the light, and the dashed line is the locus of greys and whites for each observer. (Alternatively, the dashed line is the light's vector, with no arrowhead.) Keeping in mind that there is no third dimension to these defective color spaces, we see in each case that the rules of color mixing cause a certain wavelength to match the white light, where the dashed line meets the LUM. While experimental subjects probably were not asked to match an equal energy light, the figures are consistent with the textbook "neutral points" of 494 nm for protanope, 499 nm for deuteranope, and 570 nm for tritanope[2].

This discussion is not meant to discover new results, but to show that vectorial methods can describe dichromatic color mixing using the same concepts applied to trichromats. In teaching colorimetry to students, one could omit the details of Appendix B and simply state that Figs. 12-14 show the loci of unit monochromats for dichromats. An observer with anomalous color vision could in principle be described by his LUM, but that would call for detailed data.

Camera Analysis

Maxwell-Ives Criterion. The Maxwell-Ives criterion,[22-24] says that for color fidelity a camera's spectral sensitivities must be a nonsingular transformation of those for the eye. Departures from the ideal are sometimes reduced to a figure of merit[25-27], but the concept is fundamental and a starting point for sensor design.[24]

Recall that the Locus of Unit Monochromats (LUM) is the same, independent of which transformed color matching functions are used as a starting point. Therefore, the eye's LUM expresses the combined effect of the three cone types, in an invariant form. A color camera has (at least) 3 color sensor types, so by the same logic it has its own LUM. If the camera's LUM matches that of the eye, then it sees colors like a human, meaning that it meets the Maxwell-Ives criterion. When they differ, the two LUMs define different color spaces and there is no ideal alignment. Nonetheless, the camera's LUM can be found in a convenient orientation.

The Fit First Method. In the Fit First Method[22], we find the linear combinations of the camera sensitivities which are a best fit to **Ω**. From that set of three functions, an orthonormal basis is found for the camera. Here is the computer code:

```
Rcam = RCohen(rgbSens) # 1
CamFit = Rcam*OrthoBasis # 2
GramSchmidt(CamFit, CamOmega) # 3
```

In step 1, `rgbSens` is an array whose columns are the three (or more) camera sensitivities. `RCohen()` is a short routine to apply Cohen's formula for projection matrix **R**, that is $\mathbf{R} = A[A^T A]^{-1} A^T$, so that `Rcam` is the camera's projection matrix, with A = `rgbSens`. In step 2, `OrthoBasis` is **Ω**, and left-multiplying by `Rcam` finds `CamFit`, the best fit to **Ω** using camera functions. The function call in step 3 applies Gram-Schmidt orthonormalization[28] to find `CamOmega`, an orthonormal basis for the camera functions. The Gram-Schmidt method operates on the columns of `CamFit` in sequence so that the columns of `CamOmega` show their best-fit ancestry. See App. C.

Example. Fig. 15 shows the 3 spectral sensitivities of a Nikon D1 camera[29]. These functions become the columns of array `rgbSens` in step 1. Because of the invariance of projection matrix **R**, it does not matter how the functions are normalized, or even if they are in sequence red, green, blue. Then step 2 does 3 curve fits to the 3 vectors of the orthonormal basis, Fig. 16. For example, look at the two middle curves, drawn as dash-dot. The thinner is ω_1, the human achromatic function. Step 2 finds the best fit to ω_1 by a sum of the camera functions, the thicker dash-dot curve. Functions ω_2, ω_3 are similarly fit by different combinations of camera functions. The three curve-fits are independent operations in step 2, so the best-fit functions have no necessary relationship to each other. Then in step 3 the Gram-Schmidt method yields the camera's orthonormal basis, Fig. 17, which can generate the camera's LUM, ready for comparison to the eye's.

In Fig. 18, the camera's LUM, and related 3-dimensional curves, are seen projected in the v_2-v_1 plane, while Fig. 19 shows their projection into v_2-v_3. The smooth dashed curves are the human Locus of Unit Monochromats, while solid black denotes the camera's LUM. Green arrows indicate the transition from the camera's LUM to the best fit of the human LUM by a combination of camera functions. Conceptually, the camera's LUM describes it best, but the idea of "fit first" is that the fit function (green arrowheads) is computed first, then the LUM (solid curve). Consider the projection into v_2-v_1. The camera's LUM (or fit function) tracks the human LUM remarkably well from the longest wavelengths down to about 590 nm, *in this projection*. Then a range of yellows are mapped too low in redness (v_2) and also too low in whiteness (v_1). Then a range of yellow-green to greenish wavelengths, about 570 to 510 nm, are bunched together, indicating a lack of red-green discrimination in this region. Those wavelengths are somewhat better resolved in v_2-v_3, but that does not undo the loss of red-green discrimination for a range of colors.

We could say that the camera approaches the Maxwell-Ives criterion for reds, but goes farther astray in the green. The departures between camera and eye in Fig. 19 may be harder to verbalize, but will affect pictures. This camera illustrates the method well, because of the way that it approaches human-like color matching, but with specific shortcomings. Other examples that I've worked out[22] and put on the web site, show greater overall departure from the ideal.

A precise statement of the Maxwell-Ives criterion is somewhat abstract. In effect, the camera's sensors should span the same vector space as the eye's. The Fit First method makes the abstract idea into a comparison of two graphs. The rows of `(CamFit-Orthobasis)` are vectorial error as a function of wavelength. The root-mean-square error amplitude would serve as a logical figure of merit. `Orthobasis`, otherwise called $\boldsymbol{\Omega}$, embodies the goal and `CamFit` is the camera's best effort to meet the goal, to mimic the eye as Maxwell and Ives proposed.

Discussion

The notion of orthonormal color matching functions has come up in the past[7-9,26,30,31], but each time in a narrow context. This article and recent research[1,22], have developed the idea that a single orthonormal basis can have many uses. Jozef Cohen found that color stimuli have intrinsic vector relationships, based on the fundamental metamers of the lights[6]. Emphasizing intuition and graphs rather than algebra, I reviewed the ideas of prime colors, color rendering, and projection matrix **R**, suggesting that they all relate to the overlap of receptor spectral sensitivities.[20] Then I made a discovery that ties **R** to the orthonormal basis, namely that the **R** equals the unity operator.[31] But that is almost the same as saying

$$\mathbf{R} = \boldsymbol{\Omega}\boldsymbol{\Omega}^{\mathrm{T}} \,. \tag{14}$$

Now Eq. (14) is a handy theorem[1], but in 2004, the related formula at the end of Ref. 31 was a fresh insight. See Ref. 1, App. D.

In short, pieces of a puzzle began fitting together. I recalled a discussion with a teacher who did not do color research, but mastered what was in the books. He said that "There is no diagram for color mixing."[32] Isn't that curious, one may ask, we have tristimulus vectors, but no diagrams? In about 2003, a little toying with algebra and computer graphics suggested that the invariant space could give pleasing diagrams for color mixing. A little more algebra, and discussions with

Michael Brill and Sergey Bezryadin, made clear why diagrams in XYZ space are possible but less helpful. That issue arises above under the rubric of "Color matching functions mapped to 3-vectors."

Conclusion. Six applications of vectorial color have been worked out. Analyzing them into component vectors shows how white lights work, and how artificial lights often go wrong. The method for camera analysis is suited for that exact topic: design and application of image sensors. But the algebra and concepts apply naturally to related problems. A camera is a kind of anomalous color vision observer. The fit first method could be applied to study anomalous humans or to compare color vision across species, for example. If a camera has more than 3 sensor types (with linearly independent sensitivities), the basic formalism still applies. See App. C. Cohen's ideas of vectors and invariant curves are more or less near the surface in each case.

Appendix A, The Blackbody Plane

A locus of blackbodies at constant radius in the invariant space and the vectors from the origin to those points form a gently curving surface for temperatures above 2000 K. A blackbody (BB) plane may be defined as the plane through the origin containing the color vectors of 3000 K and 10000 K blackbody. Call the unit vectors in those directions $\mathbf{u}_3$ and $\mathbf{u}_{10}$. Numerically,

$$\mathbf{u}_3 = \begin{bmatrix} 0.874 \\ 0.414 \\ 0.254 \end{bmatrix}, \mathbf{u}_{10} = \begin{bmatrix} 0.695 \\ -0.062 \\ 0.716 \end{bmatrix}, \quad \text{(A1)}$$

with an angular separation of 40.2°. To serve as axes, we need two perpendicular unit vectors in the plane of $\mathbf{u}_3$ and $\mathbf{u}_{10}$. The first can be the intersection of the BB plane and the $v_1 = 0$ plane. Points in the plane have the form $c_3\mathbf{u}_3 + c_{10}\mathbf{u}_{10}$. At the intersection with the $v_1 = 0$ plane, the first element of that vector = 0, which implies $0.874c_3 + 0.695c_{10} = 0$. Set one of the c's to unity, solve for the other, then normalize to get $\mathbf{u}_1 = [0, 0.606, -0.796]^\mathrm{T}$. The perpendicular vector is $\mathbf{u}_2 = [0.875, 0.385, 0.293]^\mathrm{T}$. If X is the matrix of axes,

$$X = [\mathbf{u}_1 \ \ \mathbf{u}_2] \quad , \quad \text{(A2)}$$

then

$$V_{\mathrm{bb}} = X^\mathrm{T} V \quad , \quad \text{(A3)}$$

is a 2-vector ready to plot. Eq. (A3) finds the components of V in the $\mathbf{u}_1$ and $\mathbf{u}_2$ directions, and discards the component normal to that plane. X is the orthonormal basis for the plane. 3-vector V is the tristimulus vector of a paint chip under a certain light, for example, with V_{bb} its projection into the plane. The plane is skew with respect to the usual axes of the invariant space, but $\mathbf{u}_1$ lies in the $v_1 = 0$ plane. It remains as an exercise for the engineer to use the projected vectors for estimating the light or for some other purpose.

The choice of 3000 K and 10000 K as starting points is convenient, not critical. Appendix B will make more clear how the blackbody locus and the chosen plane sit in the invariant color space.

Appendix B, Defining and Using a Chromaticity Diagram.

In this article and the previous one, color stimuli are graphed as vectors in the invariant 3-space, sometimes projected orthographically into a plane. The projected points or arrows are then 2-D vectors. This appendix introduces a version of chromaticity based on the invariant color space. Conceptually, color stimuli are mapped to 3-vectors from the origin, then each vector is projected to a specified plane, and the intersection is chromaticity. Chromaticity is not a vector, but a point in a plane that records a color vector's direction, losing its amplitude.

It is convenient to choose the chromaticity plane parallel to the v_2 axis and intersecting the other axes at (0.2, 0, 0) and (0, 0, 0.2) . The equations of the plane are $v_1+v_3 = 0.2$, and of course v_2 = any. If a color vector is $V = (v_1, v_2, v_3)$, then the point where it meets the plane is tV, provided that $t(v_1+v_3) = 0.2$, meaning that $t = 0.2/(v_1+v_3)$. Say that $\mathbf{r} = tV$ is the rescaled color vector. For example the vectors of the LUM, when rescaled, would trace out a spectrum locus confined to the plane, but still written as 3-vectors. To establish a coordinate system in the chromaticity plane, we find unit vectors in two directions. The first is $\mathbf{u}_1 = (0, 1, 0)$ and the second points from (0.2, 0, 0) to (0, 0, 0.2), that is $\mathbf{u}_2 = (2^{½}/2, 0, -2^{½}/2)$. If we say that **c** is a chromaticity 2-vector, then

$$\mathbf{c} = \begin{bmatrix} 0 & 1 & 0 \\ 2^{1/2}/2 & 0 & -2^{1/2}/2 \end{bmatrix} \mathbf{r}\,. \qquad \text{(B1)}$$

Boldface symbols for vectors are a reminder that the goal is a graph, not color vectors that can be further manipulated.

Fig. B1 then shows the LUM projected into the chosen plane. The dots and + signs trace a segment of the blackbody locus, with temperatures indicated. The + signs define the blackbody plane of Appendix A, shown as a dashed line.

Appendix C: Practicalities of the Fit First Method.

As presented above, the Fit First Method is complete. The three steps generate the invariant representation for the camera and compare it to the eye, as in Figs. 16-19. In this appendix, the fit first idea is augmented with a bit of computer programming, and some algebra for camera design, including the application of a four-color sensor.

Computer Code. In the Gram-Schmidt method, vectors such as color matching functions are added and subtracted to make an orthogonal set, often normalized in the process. The companion article's section on Orthonormal Functions shows the principle. The code below is in the O-matrix language,[33] similar to Matlab or even Basic. Matrix operations are part of the language. The notation `given.col(j)` refers to the j^{th} column of the array `given`. Spectral functions are referred to a common wavelength domain, with uniform steps such as 1 nm.

```
function GramSchmidt(given, orthonorm) begin
# given and orthonorm should be matrices of the same dimensions,
# usually many rows and a few columns. The column vectors
# within given will be orthonormalized to produce the column
# vectors of orthonorm.
last = coldim( given ) # coldim returns # of columns of the array
```

```
aCol = double( given.col(1) ) # 1st column, forced to double precision
SumSq = aCol'*aCol # ' denotes transpose
NormFac = 1.0d0/sqrt(SumSq) # 1.0d0 is double-precision 1.0
orthonorm.col(1) = NormFac*aCol # 1st result column
for j = 2 to last begin
      aCol = double( given.col(j) ) # again extract 1 column to work with
      for i = 1 to (j-1) begin
            dot = orthonorm.col(i)'*aCol
            aCol = aCol - dot*orthonorm.col(i)
            end # for i = 1 to (j-1)
      SumSq = aCol'*aCol
      NormFac = 1.0d0/sqrt(SumSq)
      orthonorm.col(j) = NormFac*aCol # normalize the new vector
      end # for j = 2 to last
end # end GramSchmidt()
```

The method hinges on the notion of projection. In the steps above, the expression `dot*orthonorm.col(i)` is the projection of `aCol` on the already-created basis vector `orthonorm.col(i)`.

Camera Analysis. The columns of $\mathbf{\Omega}_{cam}$ are linear combinations of the sensor functions *S*. One set may be converted to the other. Conceptually,

$$S = [|\text{red}\rangle\ |\text{green}\rangle\ |\text{blue}\rangle]\ . \quad \text{(C1)}$$

One set may be transformed to the other. For example,

$$S = \mathbf{\Omega}_{cam}\, B\ , \quad \text{(C2)}$$

where *B* is a 3×3 matrix. To find *B*, left-multiply by $\mathbf{\Omega}^{T}_{cam}$:

$$\mathbf{\Omega}^{T}_{cam}\, S = \mathbf{\Omega}^{T}_{cam}\, \mathbf{\Omega}_{cam}\, B\ . \quad \text{(C3)}$$

By orthonormality, $\mathbf{\Omega}^{T}_{cam}\, \mathbf{\Omega}_{cam} = I_{3\times3}$, implying that

$$B = \mathbf{\Omega}^{T}_{cam}\, S\ . \quad \text{(C4)}$$

Right-multiply Eq. (C2) by B^{-1} to find

$$\mathbf{\Omega}_{cam} = SB^{-1}\ . \quad \text{(C5)}$$

In application, a pixel receives a light spectrum $|L\rangle$, so that

$$\mathbf{\Omega}_{cam}{}^{T}|L\rangle = (B^{-1})^{T} S^{T}|L\rangle\ . \quad \text{(C6)}$$

Where 3-vector $S^{T}|L\rangle$ is the signal from that pixel, and $\mathbf{\Omega}_{cam}{}^{T}|L\rangle$ is that pixel as a 3-vector in the camera's invariant color space. As with any camera analysis, one does not know the detailed spectrum $|L\rangle$, so $S^{T}|L\rangle$ and $\mathbf{\Omega}_{cam}{}^{T}|L\rangle$ are explanatory names for 3-vectors, but I avoid defining more variables.

Eq. (C2) through (C5) develop a general method for converting $\mathbf{\Omega}_{cam}$ to or from any other linear combination of camera functions, denoted by *S*. Suppose that the columns of *F* are the best-fit functions from the step 2 of the fit first method. Then

$$F = \mathbf{\Omega}_{cam}\, C\ , \quad \text{(C7)}$$

where *C* is a 3×3 matrix. Then

$$C = \mathbf{\Omega}^{T}_{cam}\, F\ . \quad \text{(C8)}$$

Combine Eq. (C5) and (C7) to obtain

$$F = SB^{-1}C\ \ , \quad \text{(C9)}$$

and then by the logic of Eq. (C6), a pixel signal $S^{T}|L\rangle$ in the sensor may be converted to a pixel color $F^{T}|L\rangle$ in the best-fit color space. To summarize, matrix *S* comprises the defining camera functions, while *F* and $\mathbf{\Omega}$ are results from the fit first method. From those non-square matrices the 3×3 matrix $B^{-1}C$ is calculated for possible use in a camera.

4-band Sensor. There are camera sensors—or at least a data sheet[34]—with 4 spectral sensitivities rather than 3. In the hope that a linear combination of 4 spectral functions can approach the Maxwell-Ives ideal better than 3 functions, one may sketch out a 3-color camera using the 4-color sensor. The fit first method applies directly, but the algebra then varies a little. To start from the beginning, the columns of *S* are the camera basis, also called `rgbSens` above. By the specified names,

$$S = [|\text{yellow}\rangle\ |\text{cyan}\rangle\ |\text{magenta}\rangle\ |\text{green}\rangle] , \tag{C10}$$

but the fit first step ignores the names and sequence. The camera's projection matrix is then

$$R_{\text{cam}} = S[S^{\text{T}}S]^{-1}S^{\text{T}} . \tag{C11}$$

Eq. (C11) applies Cohen's formula, step 1 above, valid so long as the columns of *S* are linearly independent. Step 2 finds the projection of the human basis $\mathbf{\Omega}$ (3 vectors) into the 4-space of *S* . The practical meaning is 3 least-squares curve fits, no difficulty. Let *F* be the fit by the camera functions to $\mathbf{\Omega}$. Then,

$$F = R_{\text{cam}}\,\mathbf{\Omega} . \tag{C12}$$

For step 3, the camera's orthonormalized basis $\mathbf{\Omega}_{\text{cam}}$ will have only 3 vectors, derived from *F* by Gram-Schmidt. The 3 steps allow the human $\mathbf{\Omega}$ to be compared to $\mathbf{\Omega}_{\text{cam}}$ and *F* , graphically or numerically, applying the Maxwell-Ives criterion.

Again one requires a formula to find $\mathbf{\Omega}_{\text{cam}}$ from *S*, and by extension to relate signals from a pixel to the color space of the camera's orthonormalized basis. That is, we need matrix *D* such that

$$\mathbf{\Omega}_{\text{cam}} = SD . \tag{C13}$$

Assuming that the columns of *S* are linearly independent, the Moore-Penrose pseudoinverse of *S* is[35]

$$S^{+} = (S^{\text{T}}S)^{-1}S^{\text{T}} . \tag{C14}$$

Then

$$D = (S^{\text{T}}S)^{-1}S^{\text{T}}\mathbf{\Omega}_{\text{cam}} . \tag{C15}$$

From that point, the previous logic applies, but in Eq. (C9), replace B^{-1} by *D*. In Eq. (C14), *if the columns of S are orthonormal*, then $S^{\text{T}}S$ is an identity matrix and the pseudoinverse of *S* equals its transpose. With this insight, where $\mathbf{\Omega}^{\text{T}}_{\text{cam}}$ appears in Eq. (C3), it is in fact the pseudoinverse of $\mathbf{\Omega}_{\text{cam}}$.

Appendix D: Fit First Method in Relation to Other Work

Given the invariance of the LUM for each observer or camera, the fit first method applies the Maxwell-Ives criterion directly, and need not be justified by comparison to other algorithms. Still, a comparison to earlier work may be of interest.

Early TV Development. Sproson's book[36] derives from his experience during the development of color television, in 1950-1977. Color matching functions, as in Fig. 1a of the previous article, are determined by human color mixing, plus the choice of 3 primary wavelengths. A television screen has 3 primaries, so it can simulate an original scene for a human if the camera's sensitivities are those that a human would have in a color-matching experiment with those primaries. Sproson uses graphs similar to that Fig. 1a, but the detailed reasoning takes into account the system white point. A limitation of camera design is expressed by noting that the camera's sensitivity cannot have the lobes of negative sensitivity. Later, Sproson acknowledges that a linear transformation, referred to as "matrixing," can generate the negative lobes. Thus, his methods develop a sensor design from the Maxwell-Ives principle. He also uses a figure of merit based on mean departure from the ideal.

Neugebauer Quality Factor. In a preliminary step, Neugebauer's 1956 "quality factor" calculation[26] derives orthonormal CMFs U_1, U_2, U_3 , applying the Gram-Schmidt algorithm to $\{\bar{y}, \bar{x}, \bar{z}\}$ in that order. If $|f\rangle$ is one camera sensitivity, then the orthogonal function expansion is

$$|f^*\rangle = \left(\sum_{j=1}^{3} |U_j\rangle\langle U_j| \right) |f\rangle \quad . \tag{D1}$$

The summation in parentheses is in fact Matrix **R**, but the RHS as a whole is the orthonormal function expansion that Neugebauer used. Then the sum-squared error of the approximation is

$$\Delta = \langle f|f\rangle - \langle f^*|f^*\rangle = \langle f|f\rangle - \sum_{j=1}^{3} \langle U_j|f\rangle^2 \quad , \tag{D2}$$

and the quality factor is $q = 1 - \Delta/\langle f|f\rangle$. (Appendix D of the companion article may aid understanding of Eq. (D2) and Neugebauer's reasoning.)

Result q is then calculated separately for each camera sensitivity $|f_i\rangle$. It would be possible to graph $|f\rangle$ and $|f^*\rangle$ together. A limitation in the Neugebauer or Sproson work is that the camera functions $|f_i\rangle$ are not orthogonal, and one relies on the camera designer's practical sense to ensure that the 3 functions are linearly independent. Extending that thought, one could argue for orthonormalizing the camera functions just as the CMFs were orthonormalized. The idea is valid, but would lead to arbitrary dissimilar orthonormal sets, an awkward situation. By contrast, the fit first method uses the ideas of the invariant LUM, projection matrix **R**, and the specific orthonormal basis, as previously developed. The fit that is done in the fit first method is the reverse of Neugebauer's, with the human color functions approximated by the camera functions. Keeping Ω as the ideal lends intuitive sense to the graphical comparison. Orthonormal camera functions are indeed computed.

Because the *already normalized* vectors of Ω are taken as the starting point, the scale of the stated camera functions does not affect the fit first results. Neugebauer achieves a similar goal when he divides Δ by $\langle f|f\rangle$. Neugebauer finds the quality factor of the spectrum to be

$$q(\lambda) = [U_1(\lambda)^2 + U_2(\lambda)^2 + U_3(\lambda)^2]\delta\lambda \,, \tag{D3}$$

where δλ is bandwidth. Letting δλ = 1nm, this function is the distance to points on the LUM, equal to the diagonal of projection Matrix **R**.

Figure Captions
Applications of Vectorial Color
James A. Worthey

Fig. 1. The functions of the orthonormal basis.

Fig. 2. Cone stimuli, red versus green. Each dot represents a paint chip lit by D65.

Fig. 3. Dots are the same paint chips under D65. Stimuli are now expressed as v_2 versus v_1 based on Eq. (3).

Fig. 4. Dots are the same paint chips under D65. Stimuli are now expressed as X versus Y, in the usual CIE system.

Fig. 5. Color matching functions as directions in color space. The functions ω_1 , ω_2 , and ω_3 plot to the v_1, v_2, v_3 axes, which are achromatic (to the right, equal to $\bar{y}$), red-green (upward) , and blue-yellow (near to blue cones).

Fig. 6. Each arrow shows the stimuli from one color chip under blackbody lights, 5000 K at the tail, then 6500 K. Neutral chip N9.5 is a proxy for the lights, which have equal illuminance. Tristimulus vectors are projected into the v_1-v_2 plane.

Fig. 7. Similar to Fig. 6, but now v_3 is plotted versus v_1.

Fig. 8. Similar to Figs. 6 and 7, but now v_2 is plotted versus v_3. Labeled circles are a blackbody locus at v_1 = constant.

Fig. 9. The same chips are plotted in the same color space as Figs. 6-8. Now the stimuli have been projected into a blackbody plane, as defined in the text. Again, a blackbody locus is indicated.

Fig. 10. Two metameric white lights, JMW daylight and mercury vapor, are compared in their vectorial composition. The vectors are projected into v_2 versus v_1. The mercury vapor light takes a shortcut from blue to white, with less of a swing towards green (downward) and back towards red.

Fig. 11. The same color chips are plotted as in Figs 6-9, but now the lighting transition is from daylight to high-pressure mercury vapor, the lights compared in Fig. 10. This projection shows the loss of red-green contrast.

Fig. 12. Protanopic color defect considered as the absence of red receptors. The heavy solid line is the protanope’s Locus of Unit Monochromats (LUM). The thinner solid line shows the vectorial composition of the equal energy light. The dashed line is a locus of neutrals—greys and whites as seen by this color defective observer.

Fig. 13. Similar to Fig. 12, but now the subject is a deuteranope, considered to lack green receptors.

Fig. 14. Similar to Fig. 12, but now the subject is a tritanope, considered to lack blue receptors.

Fig. 15. The 3 spectral sensitivities of a Nikon D1 camera, as reported by DiCarlo, Montgomery, and Trovinger[29].

Fig. 16. Best fit to orthonormal basis Ω using the camera functions of Fig. 15. The thin curves are the human basis, while the thicker ones are the best fit. Dash-dot = achromatic, long dashes = red-green, short dashes = blue-yellow.

Fig. 17. Thinner curves are again Ω; the thicker ones are the camera's orthonormal basis.

Fig. 18. Human and camera functions projected into v_2-v_1 plane. The dashed curve is the human LUM, meaning a parametric plot of Ω. The solid curve is the camera's LUM. The heads of the short green arrows indicated the best fit of camera functions to the human LUM.

Fig. 19. Similar to Fig. 18, but the 3-dimensional curves are now projected into the v2-v3 plane.

Fig. B1. A version of chromaticity based on the invariant color space. Dashes and + signs trace a segment of the blackbody locus, with temperatures as indicated. The + signs define the blackbody plane of App. A, indicated as a dashed line.

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